Technical Comments

Comment on "Analytical Solution for Planar Librations of a Gravity Stabilized Satellite"

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REFERENCE 1 supposedly develops an approximate analytical solution to the planar librations of a gravity gradient satellite in an elliptic orbit. It will be shown that this solution is incorrect in the following ways: 1) it is not uniformly valid, 2) the equations of motion cannot be linearized, and 3) resonances are ignored.

The equation of motion of a gravity-stabilized, rigid satellite in an elliptic orbit around a spherical planet is

$$(1+e\cos\theta)\psi''-2e\psi'\sin\theta+(3/2)K\sin 2\psi=2e\sin\theta \quad (1)$$

where ψ is the pitch angle, K is an inertia parameter, e is the eccentricity, θ is the true anomaly, and primes denote differentiation with respect to θ.

Note that Eq. (1) has a forcing term of $0(e)$; therefore the oscillations of ψ will be at least of $0(e)$. Consequently a series solution in powers of e must start with e not e^0. A solution starting with e^0 will not be uniformly valid since it will not satisfy $(\psi_1/\psi_0)<\infty$ for all θ_0,θ_1. Hablani and Shrivastava[1] begin their solution at e^0; therefore, it is not uniformly valid and incorrect. Secondly, they linearize Eq. (1) and consider solutions to $0(e^3)$. Since $\sin 2\psi=2\psi-(4/3)\psi^3+\ldots$ and $\psi=0(e)$, a series solution of Eq. (1) to $0(e^3)$ must consider the ψ^3 term since it is of $0(e^3)$. This term is neglected in their solution. The fact that the series solution begins with e not e^0, and that a linear solution cannot be obtained past e^2 is stated in Beletskii,[3] which is referenced by Hablani and Shrivastava.

If it is desired to study solutions with initial conditions of 0(1) then a series solution beginning with e^0 can be sought, but the $\sin 2\psi$ cannot be linearized and the zeroth approximation will be the pendulum equation.

Thirdly, the solution of Hablani and Shrivastava has the divisors (ω_0-1) and $(\omega_0-½)$. The $\omega_0\approx 1$ is the resonance caused by the forcing term $2e\sin\theta$. The $\omega_0\approx ½$ is the well-known internal resonance which arises in the Hill type equation, Eq. (1). No mention is made of these resonances and no such limitation of their solution is stated. The correct solution will now be presented.

Using the notation of Hablani and Shrivastava, let the fast variable be θ_0 and the slow variable be θ, where

$$\theta_0=\theta$$

$$\theta_1=\sum_{n=1} e^n\tau_n\theta \quad (2)$$

where the τ_n will be determined in the analysis. Also

$$\psi(\theta)=\psi(\theta_0,\theta_1)=\sum_{n=1} e^n\psi_n(\theta_0,\theta_1) \quad (3)$$

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Substituting Eqs. (2) and (3) into Eq. (1), letting $\omega_0^2=3K$, and equating like powers of e gives

Order e^1

$$D_{00}\psi_1+\omega_0^2\psi_1=2\sin\theta_0 \quad (4)$$

Order e^2

$$D_{00}\psi_2+\omega_0^2\psi_2=-2\tau_1 D_{01}\psi_1-(D_{00}\psi_1)\cos\theta_0+2(D_0\psi_1)\sin\theta_0 \quad (5)$$

Order e^3

$$D_{00}\psi_3+\omega_0^2\psi_3=-2\tau_1 D_{01}\psi_2-(2\tau_2 D_{01}+\tau_1^2 D_{11})\psi_1 -2\tau_1(D_{01}\psi_1)\cos\theta_0-D_{00}\psi_2\cos\theta_0 +2(D_0\psi_2)\sin\theta_0+2\tau_1(D_1\psi_1)\sin\theta_0+(2/3)\omega_0^2\psi_1^3 \quad (6)$$

The first-order solution is

$$\psi_1(\theta_0,\theta_1)=A_1(\theta_1)\cos\omega_0\theta_0 +B_1(\theta_1)\sin\omega_0\theta_0+[2/(\omega_0^2-1)]\sin\theta_0 \quad (7)$$

Substitution of Eq. (7) into Eq. (5) gives

$$D_{00}\psi_2+\omega_0^2\psi_2=+2\tau_1\omega_0(A_1'\sin\omega_0\theta_0-B_1'\cos\omega_0\theta_0) +(\omega_0+2)\omega_0[A_1\cos(\omega_0+1)\theta_0+B_1\sin(\omega_0+1)\theta_0]/2 +(\omega_0-2)\omega_0[A_1\cos(\omega_0-1)\theta_0+B_1\sin(\omega_0-1)\theta_0]/2 +3/(\omega_0^2-1)\sin 2\theta_0 \quad (8)$$

Removal of the secular terms is satisfied by

$$\tau_1=0 \quad (9)$$

The solution for ψ_2 is

$$\psi_2=A_2(\theta_1)\cos\omega_0\theta_0+B_2(\theta_1)\sin\omega_0\theta_0 -(\omega_0+2)\omega_0/[4(\omega_0+½)][A_1\cos(\omega_0+1)\theta_0 +B_1\sin(\omega_0+1)\theta_0]+(\omega_0-2)\omega_0/[4(\omega_0-½)] \cdot[A_1\cos(\omega_0-1)\theta_0+B_1\sin(\omega_0-1)\theta_0] +3/[(\omega_0^2-1)(\omega_0^2-4)]\sin 2\theta_0 \quad (10)$$

Substitution of Eqs. (7), (9), and (10) into Eq. (6) gives

$$D_{00}\psi_3+\omega_0^2\psi_3=[-2\tau_2 B_1'+\sigma_1 A_1+\sigma_2 A_1(A_1^2+B_1^2)]\cos\omega_0\theta_0 +[2\tau_2 A_1'+\sigma_1 B_1+\sigma_2 B_1(A_1^2+B_1^2)]\sin\omega_0\theta_0 +\text{oscillatory terms} \quad (11)$$

where

$$\sigma_1=\frac{3\omega_0^2(1-\omega_0^2)}{8(\omega_0^2-¼)}+\frac{4\omega_0^2}{(\omega_0^2-1)^2} \quad (12)$$

$$\sigma_2=\omega_0^2/2 \quad (13)$$

Removal of the secular term gives

$$2\tau_2 A_1' = -\sigma_1 B_1 - \sigma_2 B_1 (A_1^2 + B_1^2) \quad (14)$$

$$2\tau_2 B_1' = \sigma_1 A_1 + \sigma_2 A_1 (A_1^2 + B_1^2) \quad (15)$$

This system has the first integral

$$A_1^2 + B_1^2 = k^2 = \text{constant} \quad (16)$$

The solutions of Eqs. (14) and (15) are

$$A_1 = a \cos\sigma\theta_1 - b \sin\sigma\theta_1 \quad (17)$$

$$B_1 = a \sin\sigma\theta + b \cos\sigma\theta_1 \quad (18)$$

where a and b are constants of integration and

$$\sigma = [\sigma_1 + \sigma_2 k^2]/2\tau_2 \quad (19)$$

Note that the final solution of ψ is independent of the value of τ_2; hence τ_2 can be set equal to unity.

References

[1]Hablani, H.B. and Shrivastava, S.K., "Analytical Solution for Planar Librations of a Gravity Stabilized Satellite," *Journal of Spacecraft and Rockets,* Vol. 14, Feb. 1977, pp. 126-128.

[2]Nayfeh, A.M., *Perturbation Methods,* John Wiley, New York, 1973.

[3]Beletskii, V.V., *Motion of an Artificial Satellite About Its Center of Mass,* NASA TTF-429, 1966.

Reply by Authors to K.T. Alfriend

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WE thank K.T. Alfriend for his comments on our Note.[1] The following points were raised in his comments:

1) Alfriend points out that our series solution, Eq. (23) in Ref. 1, is not uniformly valid. After examining $\psi_j(\theta)$ $(j=0,1,2)$, it is evident that for all values of θ the term $e^j\psi_j(\theta)$ is a small correction to $e^{j-1}\psi_{j-1}(\theta)$ and that the coefficient of e^j is bounded. Thus, contrary to the comment, we fail to see any nonuniformity in Eq. (23). The equation can even furnish uniformly valid eccentricity-induced oscillations.

2) The simultaneous presence of e^2 in both amplitude and frequency in Eq. (23) led us to comment that the solution was up to fourth order in eccentricity. However, we now recognize that, in fact, it is up to second order only. To establish relations of A_j, B_j $(j=1,2)$ with Θ_1, the uniformity conditions are derived from third- and fourth-order equations, respectively. This we incorrectly stated as having obtained the response up to fourth order in e. In the multiple scales method, to fully determine response up to a certain order of a small parameter, one needs to consider equations of higher order. Thus we are obliged to include equations of third order, even though response obtained was up to second order only. This supports the linearizability of Eq. (1). Further, Beletskii[2] has given solution of the *linear* equation for *nonzero* initial conditions of 0(1) using the method of Krylov-Bogolyubov. After some algebra we observe a similarity between this solution and Eq. (23), except for a few additional terms of first and second order in the latter equation. For the initial conditions and parameters of Figs. 1a and 1b of Ref. 1, the two solutions closely agree with each other and differ from the numerical response by the same amount. However, for the particulars of Fig. 1c, the Krylov-Bogolyubov method fares better. Of course, if we selected a nonlinear equation to obtain the response up to the same order it would, in principle, be more accurate than the linear response. However, in a linear regime there would be no benefits. Thus we assert that within the *linear range* of pitch angle the response recorded in Ref. 1 is correct and uniformly valid. The response arrived at by Alfriend is confined to the eccentricity-induced oscillations and is thus restrictive, although it includes nonlinearity and is expected to fare well even in the nonlinear regime.

3) Regarding resonance, a glance at any linear solution such as Eq. (23)[1] instantly reflects such points. Besides, analyses which cover these resonances are well documented (e.g., Ref. 2). As such we felt that they did not warrant any discussion.

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References

[1]Hablani, H.B. and Shrivastava, S.K., "Analytical Solution for Planar Librations of a Gravity Stabilized Satellite," *Journal of Spacecraft and Rockets,* Vol. 14, Feb. 1977, pp. 126–128.

[2]Beletskii, V.V., *Motion of an Artificial Satellite About Its Center of Mass,* NASA TTF-429, 1966, pp. 32–58.

Errata

Local Analytical Solution for Compressible Turbulent Boundary Layers

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EQUATION (19) should read as

$$C = 0.0176\left(\frac{T_{\rm ref}}{T_e}\right)^{-1+\omega i}\left(\frac{T_e}{T_0}\right)^{-\left(\frac{1}{\gamma-1}-\omega\right)i}\left(\frac{\rho_0\mu_e}{\rho_e\mu_0}\right)^{-i}$$

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